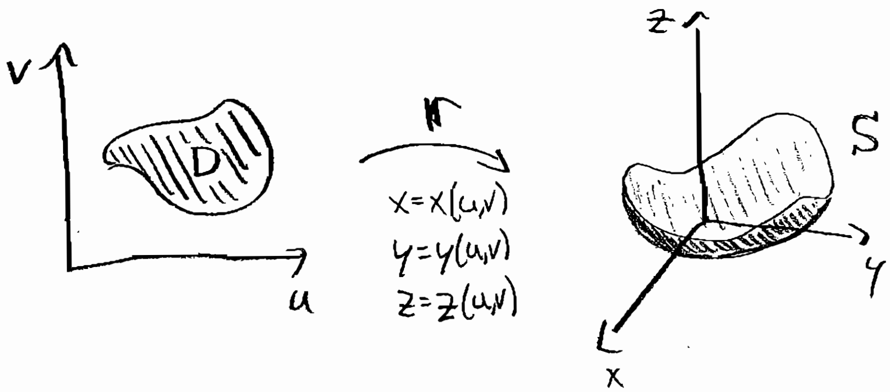
Parametric Surfaces and Tangent Planes

[Andrew Critch, Math 53]

A parametric surface in $\mathbb{R}^3$ is a surface S expressed as the range of a function from a patch of $\mathbb{R}^2$ (say the u,v plane) into $\mathbb{R}^3$ (say x,y,z space):

$$\mathbf{r} : \left(\mathbb{R}^2_{u,v}\right) \to \mathbb{R}^3_{x,y,z}, \quad S = \text{range}(\mathbf{r})$$



If D only covers S once (the way $[0,2\pi]$ only covers the circle once under $x=\cos t$, $y=\sin t$), I'll indicate this by

$$\mathbf{r} : D \xrightarrow{1} S.$$

(This is very much like a change of variables!)

Example: Say $S = \{x^2+y^2+z^2=1\}$ in $\mathbb{R}^3_{x,y,z}$. If we let

$$\mathbf{r}(\phi,\theta) = \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} = \begin{bmatrix} x(\phi,\theta) \\ y(\phi,\theta) \\ z(\phi,\theta) \end{bmatrix}, \text{ and } D = \boxed{\begin{Bmatrix} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{Bmatrix}} \subseteq \mathbb{R}^2_{\phi,\theta}$$

then $S = \text{range}(\mathbf{r})$, and in fact $\mathbf{r} : D \xrightarrow{1} S$.

($\mathbf{r}$ parametrizes S "once" with D). //

Example: Say $S = \{z = e^{x+y}\}$ in $\mathbb{R}^3_{x,y,z}$. If we let

$\boxed{x = u,\ y = v,\ z = e^{u+v}}$, i.e. $\mathbf{r}(u,v) = \boxed{\begin{pmatrix} u \\ v \\ e^{u+v} \end{pmatrix}}$, then

$S = \text{range}(\mathbf{r})$, and in fact $\mathbf{r} : \mathbb{R}^2_{u,v} \xrightarrow{1} S$. //

any graph can by parametrized in a similar way!

Given equations for a parametric surface, how can you visualize it? One idea I like to use is that "implied equations define supersets." *

*(also useful for plotting implicit curves!)

Example: Say $\begin{array}{l} x = u^2 + v \\ y = u^2 - v \\ z = 1 - 2u^2 \end{array}$ parametrizes S.

Adding these equations shows that $\boxed{x + y + z = 1}$.

This implies that S is inside the plane $\{x + y + z = 1\}$ — called a "superset" of S, because S is inside it.

It is not the whole plane, since $z = 1 - 2u^2 \leq 1$

So S is also inside the half-space $z \leq 1$

(exercise: convince yourself that $S = \{x+y+z=1\} \cap \{z \leq 1\}$)

- Stewart 16.6 explains another important visualization technique: grid curves (setting u or v to a constant).

- He also shows many important examples of how to parametrize a given surface, which can be quite tricky!

Just like any (differentiable) map between spaces,

$\mathbf{r} : \mathbb{R}^2_{u,v} \to \mathbb{R}^3_{x,y,z}$ has a Jacobian matrix.

If we write $\mathbf{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$, then

$$J_{\mathbf{r}} = \boxed{\begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}}.$$

We also write: $\mathbf{r}_u = \boxed{\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}}$ and $\mathbf{r}_v = \boxed{\begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}}$

And of course $J_{\mathbf{r}}$ satisfies a linear approximation formula.
Writing $\mathbf{u}$ and $\mathbf{u}_0$ for (u,v) and (u_0,v_0), it is:

$$\boxed{\mathbf{r}(\mathbf{u}) - \mathbf{r}(\mathbf{u}_0) \overset{\mathbf{u}\to\mathbf{u}_0}{\sim} J_{\mathbf{r}}(\mathbf{u}_0)\langle \mathbf{u} - \mathbf{u}_0 \rangle}$$

Example: Say $\mathbf{r}(u,v) = \begin{pmatrix} u^2+v \\ u^2-2v \\ 2-2u^2 \end{pmatrix}$. $J_{\mathbf{r}} = \boxed{\begin{bmatrix} 2u & 1 \\ 2u & -2 \\ -4u & 0 \end{bmatrix}}$.

To approximate $\mathbf{r}$ near the input $(1,2)$:

$J_{\mathbf{r}}(1,2) = \boxed{\begin{bmatrix} 2 & 1 \\ 2 & -2 \\ 4 & 0 \end{bmatrix}}$, $\mathbf{r}(1,2) = \boxed{\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}}$, thus

$$\mathbf{r}(u,v) \overset{\boxed{\substack{u\to 1 \\ v\to 2}}}{\sim} \mathbf{r}(1,2) + J_{\mathbf{r}}(1,2)\boxed{\begin{pmatrix} u-1 \\ v-2 \end{pmatrix}} = \boxed{\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & -2 \\ -4 & 0 \end{bmatrix}\begin{pmatrix} u-1 \\ v-2 \end{pmatrix}}$$

$$= \boxed{\begin{pmatrix} 3+2(u-1)+1(v-2) \\ -3+2(u-1)-2(v-2) \\ 0-4(u-1)+0(v-2) \end{pmatrix}} = \underbrace{\boxed{\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}u + \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}v}}_{\text{Call this "}\tilde{\mathbf{r}}\text{"}}$$

Example cont'd | To find the tangent plane to the surface $S = \text{range}(\mathbb{r})$ (given by equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbb{r}(u,v)$) at the point $\mathbb{r}(1,2)$, we replace $\mathbb{r}$ by $\tilde{\mathbb{r}}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tilde{\mathbb{r}} = \boxed{\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} u + \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} v}.$$

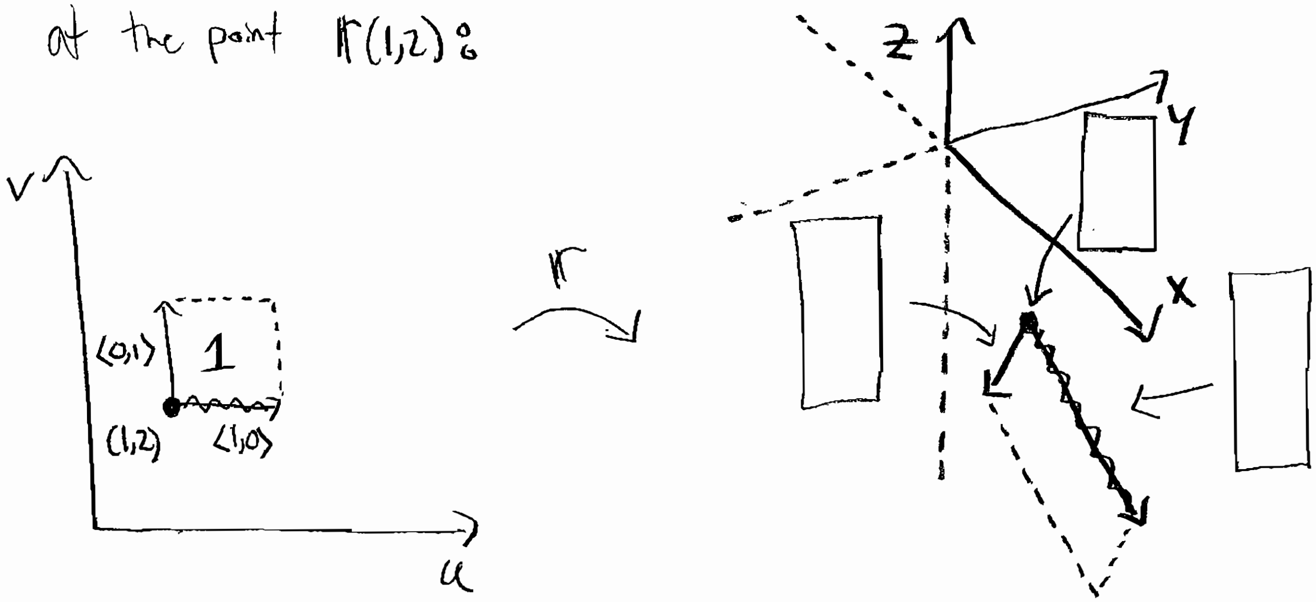
This is a parametrization of the tangent plane*, and as usual, a parametrization easily shows us vectors parallel to the plane: $\mathbb{r}_u(1,2) = \boxed{\left\langle \begin{matrix} 2 \\ 2 \\ -4 \end{matrix} \right\rangle}$ and $\mathbb{r}_v(1,2) = \boxed{\left\langle \begin{matrix} 1 \\ -2 \\ 0 \end{matrix} \right\rangle}$. To get an implicit equation, we can "eliminate parameters", or just find a normal vector $\mathbb{r}_u(1,2) \times \mathbb{r}_v(1,2) = \boxed{\left\langle \begin{matrix} -8 \\ -4 \\ -6 \end{matrix} \right\rangle}$. We just found a (non-unit) normal vector to S at $\boxed{\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}}$!

* Just like in line parametrizations, using any any other point on the plane as a "starting point" will not change the range, so...

... $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} u + \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} v}$ also parametrizes the plane, even though it is not a linear approximation for the function $\mathbb{r}(u,v)$ at $(1,2)$.

textbook method ↑ (fast)

Example Cont'd | Just like in a change of variables, $\tilde{\mathbf{r}}$ sends an (ordered) unit square at the input point $(1,2)$ to the (ordered) parallelogram $\mathbf{r}_u(1,2), \mathbf{r}_v(1,2)$ at the point $\mathbf{r}(1,2)$:



This parallelogram has a positive area of

$\boxed{} = \boxed{} = \boxed{}$.

In other words, this is the (positive) stretch factor for $\mathbf{r}$ itself at the input $(1,2)$!

Now we're finally ready for surface integrals :)